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Punctured discs on the square lattice

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Abstract. We study the perimeter length of self-avoiding surfaces on the square lattice. We prove that surfaces with area n, containing h boundary components, have mean perimeter length of order n. We arrive at this result by studying the incidence of 4-cycles in two-dimensional site animals.

1. Introduction

Self-avoiding surfaces have received considerable attention in the literature. Glaus (1986, 1988) obtained numerical evidence that surfaces on the simple cubic lattice, homeomorphic to discs, may have the same critical exponents as lattice animals. Since lattice trees are thought to have the same critical exponents as lattice animals (see, for instance, Gaunt *et al* (1982)), this implies that discs in three dimensions are in the same universality class as lattice trees. This suggests that discs are highly ramified objects and, in this paper, we present some rigorous results which establish this in two dimensions.

We write $\mathscr{Z}^2$ for the two-dimensional hypercubic lattice. A *self-avoiding surface* on $\mathscr{Z}^2$ is a connected set of plaquettes (elementary unit squares), joined along common edges so that each edge belongs to either one or two plaquettes and, if precisely two plaquettes are incident on a vertex, then they must share an edge. The edges incident on only one plaquette form disjoint closed curves in $\mathscr{Z}^2$, which we call the boundary components of the surface. Let $\mathscr{S}_n(h)$ be the set of all self-avoiding surfaces in $\mathscr{Z}^2$ with h boundary components and n plaquettes. Let the cardinality of $\mathscr{S}_n(h)$ be $s_n(h)$.

The simplest examples of self-avoiding surfaces in two dimensions are discs, in the set $\mathscr{S}_n(1)$. The boundary curve of a disc is a polygon of length m and enclosing area n, so that we may think of discs as polygons of area n. Polygons on the square lattice have been studied for many years and Hiley and Sykes (1961) and Enting and Guttmann (1989) have enumerated polygons by perimeter and by area.

This paper is organised as follows. In section 2 we define surface animals. These are site animals dual to self-avoiding surfaces. We point out that the perimeter of surfaces with a fixed number of boundary components is related to the numbers of 4-cycles in the dual animals. We then consider the properties of the 4-cycles in site animals (as well as in surface animals). We prove that there are two classes of 4-cycles. The first class are those which can be deleted from the animal by removing a single vertex without disconnecting the animal. We call these *ordinary* cycles. The second class of 4-cycles are those which we cannot remove without disconnecting the animal;

these we call *solitary* cycles. We also prove that solitary cycles are truly solitary, i.e. if two 4-cycles shares an edge, then neither can be solitary.

In section 3 we study the incidence of ordinary 4-cycles in surface animals. In particular, if $t_n^k(c)$ is the total number of site animals, with precisely c ordinary 4-cycles, dual to a self-avoiding surface with n plaquettes and $h = k + 1$ boundary components, then we use the results of Madras *et al* (1988) to find bounds on the function

$$\psi(\varepsilon) = \limsup_{n\to\infty} (t_n^k(\lceil \varepsilon n \rceil))^{1/n} \tag{1.1}$$

and show that $\psi(\varepsilon)$ goes to zero as ε goes to 1. We then use this result to prove the existence of a positive constant z such that

$$\liminf_{n\to\infty} \frac{\langle v_n^h \rangle}{n} \geq z > 0 \tag{1.2}$$

where $\langle v_n^h \rangle$ is the mean perimeter length of a self-avoiding surface with h boundary components and area n. We conclude the paper with a few comments in section 4.

2. Surface and lattice animals

Let $\sigma_n \in \mathscr{S}_n(h)$. We can consider a lattice animal τ_n dual to σ_n by letting each vertex of τ_n on the dual lattice (to $\mathscr{L}^2$) correspond to a plaquette on σ_n. (We simply replace every plaquette on σ_n by a vertex at its midpoint.) If two plaquettes share a common edge, then the two vertices in the dual animal are connected by an edge. We call τ_n a *surface animal*, and these form a subset of all site animals. Not all site animals are surface animals, since the patterns in figure 1 cannot occur in a surface animal because that would violate the self-avoiding condition.

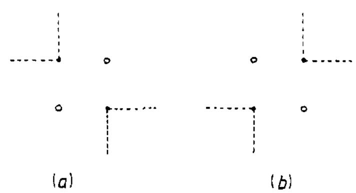

Figure 1. These patterns are forbidden in surface animals, since they do not correspond to self-avoiding surfaces.

A typical surface animal will contain cycles. The smallest cycles will be 4-cycles, but larger cycles will in general also occur. An m-cycle, $m > 4$, is *contractible* if it is composed of 4-cycles. We define all 4-cycles to be contractible. Therefore, no non-contractible cycle can be a 4-cycle. The fundamental homotopy group of the surface $\sigma_n \in \mathscr{S}_n(h)$ has $(h - 1)$ generators. These are non-contractible closed curves on σ_n. If we choose these generators so that they pass through the midpoints of every plaquette that they visit on σ_n, and always pass from one plaquette to the next through an edge (never through a vertex) then, under the duality transformation, they will map to non-contractible cycles on the dual animal τ_n. If two non-contractible cycles on τ_n are in the image of the same homotopy class of a generator on σ_n, then we call them *equivalent* (it is easily seen that this is an equivalence relation on the non-contractible cycles). These definitions divide the non-contractible cycles on τ_n into equivalence classes, which we call *independent cycles.* Since the map from the homotopy classes of the generators on σ_n to the set of independent cycles is a bijection, the number of independent cycles is $(h-1)$. Let $\mathscr{T}_n^k$ be the set of surface animals with k independent cycles and n vertices, dual to the surfaces in the set $\mathscr{S}_n(k+1)$. The number of 4-cycles in the animals in $\mathscr{T}_n^k$ is independent of k.

Let $\sigma_n \in \mathscr{S}_n(h)$ and let $v(\sigma_n)$ be the perimeter length of σ_n (the total number of edges on the boundary components of σ_n). Evidently, $v(\sigma_n) = d_3+2d_2+3d_1$, where d_i is the number of ith degree vertices on the surface animal τ_n dual to σ_n. For any animal, the number of cycles c' is given by

$$2c' = 2 - d_1 + d_3 + 2d_4. \tag{2.1}$$

Hence, by using the fact that $n = \sum_i d_i$, we find that

$$v(\sigma_n) = 2 + 2n - 2c'. \tag{2.2}$$

But $c' = h - 1 + c$, where c is the number of 4-cycles in τ_n, and h the number of boundary components on the dual surface σ_n. Thus

$$v(\sigma_n) = 2n - 2h - 2c + 4. \tag{2.3}$$

We need the following definitions.

Definition 2.1. Let $\mathscr{V}$ be a set of vertices in $\mathscr{Z}^2$. The *top* vertex and the *bottom* vertex of $\mathscr{V}$ are found through a lexicographic ordering of the vertices in $\mathscr{V}$.

Definition 2.2. Let o be a 4-cycle in $\tau_n \in \mathscr{T}_n^k$. If there exists a vertex on o such that deleting the vertex will remove o without disconnecting τ_n, then we call o an *ordinary* cycle. If o is not ordinary, then we call it a *solitary* cycle.

Definition 2.3. The *cycle set* $\mathscr{C}$ of an animal $\tau_n \in \mathscr{T}_n^k$ is the set of all vertices in τ_n which are incident on an ordinary 4-cycle. If $\mathscr{C}$ is empty, then τ_n has no ordinary 4-cycles.

We now prove that every solitary cycle is one of the two cases in figure 2. In particular, if two 4-cycles share an edge, then they are both ordinary. In addition, all 4-cycles not sharing an edge with another are ordinary, unless they are one of the cases in figure 2.

Lemma 2.4. Let $\tau_n \in \mathscr{T}_n^k$. If any two 4-cycles in τ_n share an edge, then they are both ordinary. Moreover, the only solitary 4-cycles in τ_n are those in figure 2.

Proof. We give a direct proof by showing that if we cannot delete a 4-cycle, then it is one of the cases in figure 2. Let $\mathscr{C}$ be the cycle set of τ_n, and let t be the top vertex of $\mathscr{C}$. Let the two orthogonal unit vectors in $\mathscr{Z}^2$ be e_1 and e_2.

By definition, t is incident on an ordinary 4-cycle, and is a vertex of degree 2, 3 or 4. If t is of degree 2, then we can delete it, erasing the 4-cycle, so we suppose that t is of degree 3 or 4.

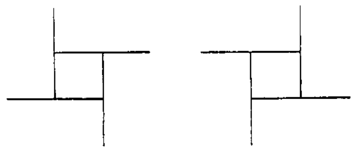

Figure 2. The two solitary 4-cycles in two dimensions.

Suppose that t is a vertex of degree 3, and without loss of generality, suppose that $(t+e_1)$ is in τ_n (while $(t+e_2)$ is not). Then $(t+e_1-e_2)$ cannot be in τ_n. Consider the vertex $(t-e_2)$; if $(t-2e_2)$ is not in τ_n, then it is of degree 2, and we can delete it, removing the 4-cycle incident on t, so suppose that $(t-2e_2)$ is in τ_n. If $(t-e_1-2e_2)$ is also in τ_n, then we can also delete $(t-e_2)$, erasing two 4-cycles, one of which is incident on t, without disconnecting the animal. So suppose that $(t-e_1-2e_2)$ is not in τ_n. If $(t-2e_1-e_2)$ is also not in τ_n then the vertex $(t-e_1-e_2)$ is of degree 2, so we can delete it to erase the 4-cycle incident on t, so let $(t-2e_1-e_2)$ be in τ_n. If $(t-2e_1)$ is also in τ_n, then we can delete the vertex $(t-e_1-e_2)$, erasing two 4-cycles without disconnecting τ_n, so suppose that $(t-2e_1)$ is not in τ_n. If $(t-e_1+e_2)$ is also not in τ_n, then the vertex $(t-e_1)$ is of second degree, and we can erase the 4-cycle, so let $(t-e_1+e_2)$ be in τ_n. We have now considered every vertex on the 4-cycle incident on t, and showed that if we cannot erase it, then it is the 4-cycle in figure 2(a). But this is a contradiction, since $\mathscr{C}$ contains only vertices incident on ordinary cycles, so at some stage in this construction we must succeed in erasing the 4-cycle incident on t. Therefore we cannot have the 4-cycle in figure 2(a) incident on the vertex t. By exchanging the unit vectors e_1 and e_2 in this argument, we find that 4-cycles like that in figure 2(b) cannot be be incident on t. Therefore, if t is a 3-degree vertex, then the only 4-cycles on which it cannot be incident are those in figure 2. Note that these are truly solitary cycles, since they do not share an edge with another 4-cycle.

All that is left to do is to check the case when t is a 4-degree vertex. Then the vertices $(t+e_1)$ and $(t+e_2)$ are both in τ_n, and $(t-e_1+e_2)$ and $(t+e_1-e_2)$ are not in τ_n. The construction is now the same as above. If $(t-2e_2)$ is not in τ_n, then $(t-e_2)$ is a second-degree vertex and we can delete the 4-cycle incident on t, so let $(t-2e_2)$ be in τ_n. If $(t-e_1-2e_2)$ is also in τ_n, then we can delete $(t-e_2)$, erasing two 4-cycles without disconnecting the animal, so let $(t-e_1-2e_2)$ not be in τ_n. Then $(t-2e_1-e_2)$ must be in τ_n, or $(t-e_1-e_2)$ is of degree 2, and we can delete the 4-cycle. If $(t-2e_1)$ is also in τ_n, then we can delete $(t-e_1-e_2)$, erasing two 4-cycles without disconnecting τ_n, so let $(t-2e_1)$ not be in τ_n. But then $(t-e_1)$ is of degree 2, so we can delete it, erasing the 4-cycle incident on t. Therefore, if t is a 4-degree vertex, then we can always erase the 4-cycle incident on it by deleting a single vertex.

To see that the cycles in figure 2 cannot be ordinary we argue as follows. $\mathscr{C}$ is a finite set, and suppose that it contains one of the solitary cycles in figure 2. By the arguments above, this cycle cannot be incident on t, the top vertex of $\mathscr{C}$, so suppose that it is somewhere else in $\mathscr{C}$. We then apply the construction above, every time that we delete a vertex to erase the ordinary cycle incident on t, the cardinality of $\mathscr{C}$ is reduced by one, while we find a new top vertex. Thus, after a finite number of deletions, the new top vertex will be incident on the solitary cycle, which we cannot remove. This is a contradiction (by the definition of $\mathscr{C}$).

Let $\mathcal{T}_n^k(c)$ be the set of surface animals dual to surfaces in $\mathcal{S}_n(k+1)$ with precisely c ordinary 4-cycles and any number of solitary cycles. Let the cardinality of $\mathcal{T}_n^k(c)$ be $t_n^k(c)$.

3. 4-cycles and perimeter length

Let $t_n^k(\lceil \varepsilon n \rceil)$ be the total number of surface animals with k independent cycles, n vertices, precisely $\lceil \varepsilon n \rceil$ ordinary 4-cycles and any number of solitary cycles. Define

$$\psi^k(\varepsilon) = \limsup_{n\to\infty} \left(t_n^k(\lceil \varepsilon n \rceil)\right)^{1/n} \tag{3.1}$$

for all ε in $[0,1)$. It is a tedious but straightforward exercise to show that $\lim_{n\to\infty} \left(t_n^k(\lceil \varepsilon n \rceil)\right)^{1/n}$ exists, is log–concave in $(0,1)$ and continuous in $[0,1)$. However, since we do not require that result here, we only consider (3.1) and prove that is enough to give us our desired results. To continue, we need a result for *edge* animals due to Madras *et al* (1988).

Lemma 3.1 (Madras et al 1988). Let $a_n(\lceil \varepsilon n \rceil)$ be the number of edge animals with $\lceil \varepsilon n \rceil$ cycles and n vertices. Then the limit

$$\lim_{n\to\infty} \left(a_n(\lceil \varepsilon n \rceil + l)\right)^{1/n} = \lim_{n\to\infty} \left(a_n(\lceil \varepsilon n \rceil)\right)^{1/n} = \phi(\varepsilon)$$

exists for any positive fixed integer l, is log-concave in $(0,1)$ and continuous in $[0,1)$. Moreover

$$\lim_{\varepsilon\to 1^-} \phi(\varepsilon) = 1.$$

Lemma 3.2.

$$\lim_{\varepsilon\to 1^-} \psi^k(\varepsilon) = 1.$$

Proof. $\psi^k(\varepsilon)$ is bounded from above by $\phi(\varepsilon)$. We prove this by constructing an injection from surface animals with c ordinary 4-cycles and k independent cycles into the set of edge animals with $c+k$ cycles. This is accomplished by removing a particular edge, for example the top edge, from each solitary cycle of the surface animal (see figure 3). The lemma then follows from lemma 3.1.

Lemma 3.3. $\psi^k(0) \geq \sqrt{\mu}$, where $\mu > 1$ is the growth constant of self-avoiding walks.

Proof. That $\psi^0(0)$ is bounded from below by $\sqrt{\mu}$ is seen by an injection from self-avoiding walks with step length 2 into $t_n^0(0)$ (if n is odd, then we consider walks of length $n-1$, and concatenate a single vertex on the top vertex of each walk).

To prove the result for $\psi^1(0)$, we consider polygons of step length 2 and length n (n even) or $(n-1)$ (n odd). If n is odd, we concatenate a single vertex onto every polygon. For $\psi^k(0)$, $k > 1$, we consider the set of polygons with step length 2 and

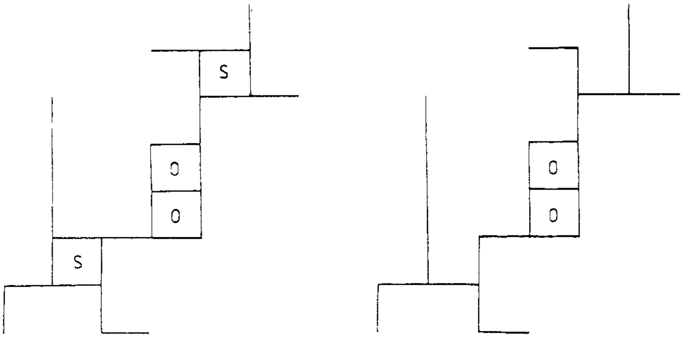

Figure 3. A surface animal with two ordinary cycles (O) and two solitary cycles (S) and its associated edge animal obtained by deleting the top edge in each solitary cycle.

length $(n+8-8k)$ (n even) or $(n+7-8k)$ (n odd). We then concatenate $(k-1)$ 8-cycles onto each polygon (and an extra vertex if n is odd).

Since $\psi^k(\varepsilon)$ is not identically equal to 1 in $[0,1)$, but goes to 1 as $\varepsilon \to 1$, we have the following result.

Corollary 3.4. There exists an ε_o in $[0,1)$ such that

$$0 \leq \varepsilon_o = \max_{\varepsilon}\{\psi^k(\varepsilon) = \max_{\alpha}\{\psi^k(\alpha)\}\} < 1.$$

This result implies that in the $n \to \infty$ limit all except exponentially few surface animals will have at most $\lceil \varepsilon_o n \rceil$ ordinary 4-cycles, provided that we fix the number of independent m-cycles. Suppose that an animal has $\lceil \varepsilon n \rceil$ ordinary 4-cycles. Then the maximum number of solitary cycles (c_s) it may have (since each solitary cycle takes four vertices) is

$$c_s \leq \tfrac{1}{4}(n - \lceil \varepsilon n \rceil - 2\lceil \sqrt{\lceil \varepsilon n \rceil} \rceil). \tag{3.2}$$

Thus, from (2.3), the perimeter length of a surface with h boundary components dual to a surface animal with $\lceil \varepsilon n \rceil$ ordinary 4-cycles is bounded from below by

$$v_n^h \geq \frac{3}{2}(n - \lceil \varepsilon n \rceil) + \lceil \sqrt{\lceil \varepsilon n \rceil} \rceil - 2h + 4 = w_n^h. \tag{3.3}$$

We thus find the following theorem.

Theorem 3.5. There exists a constant $z > 0$ in two dimensions such that

$$\liminf_{n\to\infty} \frac{\langle v_n^h \rangle}{n} \geq z > 0$$

where $\langle v_n^h \rangle$ is the mean perimeter length of a surface with n plaquettes and h boundary components.

Proof. Evidently

$$\liminf_{n\to\infty} \frac{\langle v_n^h \rangle}{n} \geq \liminf_{n\to\infty} \frac{\sum_{i=0}^{n-1} (w_n^h/n) t_n^{h-1}(i)}{\sum_{i=0}^{n-1} t_n^{h-1}(i)}$$

as we see from equation (3.3). Both the denominator and the numerator have n terms, each term growing exponentially in n. The terms growing fastest are those with largest growth constant, that is, $\psi^k(\varepsilon_o)$, where $\varepsilon_o < 1$. Thus

$$\liminf_{n\to\infty} \frac{\langle v_n^h \rangle}{n} \geq \tfrac{3}{2}(1 - \varepsilon_o) = z > 0.$$

Obviously, since $2n \geq v_n^h$ we have

$$2 \geq \limsup_{n\to\infty} \frac{\langle v_n^h \rangle}{n} \geq \liminf_{n\to\infty} \frac{\langle v_n^h \rangle}{n} \geq z > 0. \tag{3.4}$$

If we imply equation (3.4) with the relation $\sim$, then we can write

$$\langle v_n^h \rangle \sim n. \tag{3.5}$$

4. Conclusions

We have established rigorously that the perimeter of self-avoiding surfaces (with a fixed number of boundary components) grows proportionally to the area of the surface, where proportionally means in the sense of equation (3.4).

The results in section 3 agree with those of Enting and Guttmann (1989) obtained by enumerating polygons by area and by perimeter. The message from these results is that self-avoiding surfaces in two dimensions are highly ramified objects; every plaquette on the surface will have an edge on the perimeter with positive probability. These results do not imply that a surface will become disconnected with probability one (in the large-n limit) if we delete a plaquette at random. On the contrary, a pattern theorem for discs (Madras 1989) implies that, at least in the case $h = 1$, $\varepsilon_o > 0$ (that is, $\psi^1(\varepsilon)$ is strictly increasing in some interval $[0, 0 + \delta)$, $\delta > 0$). This implies that

$$\limsup_{n\to\infty} \frac{\langle v_n^1 \rangle}{n} < 2. \tag{4.1}$$

Therefore, with positive probability, a plaquette will have less than two edges on the perimeter of the disc, and if we delete it, we will not necessarily disconnect the animal. We expect these arguments to apply to surfaces with more than one boundary component too.

The existence of the limit $\lim_{n\to\infty} \langle v_n^h \rangle / n$ was not established in this study. This is an interesting problem, but may be very hard to prove. Rigorous bounds on ε_o (theorem 3.5); would be useful, especially for comparison to numerical simulations.

Acknowledgment

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